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Kelly trading when asset prices have jumps

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Abstract

If asset prices have no jumps it is known that the growth optimal Kelly strategy lies on the local efficient frontier, that is has maximal instantaneous Sharpe ratio. In this paper we show that, when asset prices have jumps, this property ceases to hold. However, the discrepancy is small for jumps with magnitude less than 25%. Our results further indicate that a Kelly trader fears market crashes but favors stock picking more than a maximal Sharpe trader. We also explain why Merton's approach to jump modeling is not well suited to study the risk of bankruptcy associated with leverage.

Keywords Kelly criterion · Maximal Sharpe · Jumps · Portfolio theory

JEL Classification G11 · G12

1 Introduction

If asset prices have no jumps, it is known (e.g. Bermin and Holm (2021, 2023)) that the growth optimal Kelly strategy lies on the local efficient frontier; that is, it has a maximal instantaneous Sharpe ratio (Nielsen and Vassalou (2004)). In this paper, we show that when asset prices have jumps, this property ceases to hold. We model asset prices as jump-diffusions, using the framework of a marked point process (Jacod and Shiryaev (2003); Björk (2021)), and provide extensive details to highlight the ease with which it can be applied. We also compare the chosen approach with that of Merton (1976) and find that his approach can largely underestimate the risk of bankruptcy (when high leverage is applied).

Within the original (single-period) expected utility framework (von Neumann and Morgenstern (1947)), the mean-variance model (Markowitz (1952)) plays a central role; especially among practitioners. It is not a wild guess that the popularity of the mean-variance model is due to the resulting strategy having a maximal Sharpe ratio (Sharpe (1966, 1994)), rather than corresponding to a quadratic utility function. The

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question arises: If we do not believe in the (single-period) quadratic utility specification, can we still believe that maximizing the Sharpe ratio is a reasonable objective for asset allocation?

A few years after Markowitz's publication, Kelly (Kelly (1956)) developed a novel approach to optimal capital formation. Kelly stressed that the important quantity to look at is the excess logarithmic return (rather than the excess return) and argued: "The reason has nothing to do with the *value* function which [the investor] attached to his money, but merely with the fact that it is the logarithm which is additive in repeated bets and to which the law of large numbers applies." The corresponding trading strategy is known as the *Kelly criterion* or, as we call it, the growth optimal Kelly strategy. Subsequent to Kelly's publications, many authors (e.g. Hakansson (1971) and Elton and Gruber (1974), to name just a few) investigated which utility functions remained stable when extending the expected utility approach from a single period to multiple periods. The analysis showed that essentially only logarithmic and power utilities were feasible choices, thus supporting (in a way) the approach of Kelly as his could be cast into an extended multi-period expected utility framework with a logarithmic utility function. In Goll and Kallsen (2003) it was finally shown, within a general jump-diffusion setting, that the growth optimal Kelly strategy is identical to an optimal strategy in the sense of expected utility for the logarithmic utility function. Over the years, the theoretically oriented debate has been fierce, with, in particular, Samuelson taking a strong objection to Kelly's approach (see, for instance, MacLean et al. (2011) and Ziemba (2015) for a historical recount).

History aside, Kelly's approach showed that investors make money by rebalancing their portfolios in two ways: both from directional changes and from the volatility of the assets traded. Bermin and Holm showed in a sequel of papers (Bermin and Holm (2021, 2023)) that when asset prices have no jumps, Kelly's approach leads to the maximization of the instantaneous Sharpe ratio (Nielsen and Vassalou (2004)) throughout time. In effect, the authors went as far as calling such maximal Sharpe strategies Kelly strategies. The purpose of this paper is to show that one must make a distinction between Kelly strategies and maximal Sharpe strategies when the primary assets are modeled as jump-diffusions. Henceforth, we refer to Kelly strategies as those strategies that are instantaneously collinear to the growth optimal Kelly strategy; thus preserving the latter's instantaneous Sharpe ratio. In other words, what we call Kelly strategies is a slight generalization of the fractional Kelly strategies in MacLean et al. (1992). Furthermore, as shown in Davis and Lleo (2013), the Kelly strategies correspond closely, but not perfectly, to the optimal expected power utility strategies. With that being said, we provide evidence that the distinction is mainly relevant for highly volatile trading strategies where positive jumps dominate. For additional information on optimal expected utility strategies; see, for example, Callegaro et al. (2006); Björk et al. (2010) for power utility, and Goll and Kallsen (2003) for logarithmic utility.

To summarize our findings, we return with an answer to the very first question raised: yes, maximizing the Sharpe ratio (more precisely, the instantaneous Sharpe ratio) is a reasonable objective for asset allocation. Thus, the practitioners of the mean-variance model have been mostly correct, but in order to realize this we have had to accept intra-period trading and replace the (single-period) quadratic utility function with a (multi-period) logarithmic or power utility function. We further stress that the discrepancy between maximal Sharpe strategies and Kelly strategies is not related to the distributions of the primary assets, but rather to the very nature of the evolution of these assets. For example, let $P(t)$ denote the price (at time t) of an asset with distribution function $F(x; t) = \mathbb{P}(P_t \leq x)$. Then, we can always find a diffusion model that generates the same distribution simply by setting $P_t = F^{-1}(N(W_t/\sqrt{t}); t)$, where N denotes the distribution function of a standard Gaussian random variable, and W is a one-dimensional Brownian motion. This means that the observed discrepancy has nothing to do with the deviation of, say, log-normality for asset returns. Instead, it is a consequence of the local properties; namely whether the asset prices have jumps or not. The fact that a Kelly trader can take advantage of the jumps is, at least, theoretically interesting.

This paper is organized as follows. In Sect. 2, we present the jump-diffusion model. The jumps are introduced via a marked point process (e.g. Jacod and Shiryaev (2003); Björk (2021)), while the diffusion comes from a standard Brownian motion. We elaborate on how to include the randomness of the jumps in the instantaneous covariance matrix in a smooth fashion; following closely the work by Björk and Slinko (2006) and Christensen and Platen (2007). In Sect. 3 we define and characterize the maximal Sharpe and Kelly strategies. Subsequently, in Sect. 4, we present two case studies to highlight the differences between the trading strategies. Our results indicate that a Kelly trader fears market crashes but favors stock picking more than a maximal Sharpe trader. In Sect. 5, we investigate Merton's approach to jump modeling (Merton (1976)) and show that this approach can either largely overestimate or underestimate the risk of bankruptcy when applied to leveraged portfolios. Instead, we provide a framework for studying default risk based on discrete time trading and worst-case jump analysis. In Sect. 6, we look closer at what volatility means in the presence of jumps. Our findings indicate that it is easier to approximate a Kelly strategy than a maximal Sharpe strategy using a model without jumps. Section 7 concludes.

2 Modeling the market

We consider a capital market consisting of a number of primary assets $(P_0, P_1, \ldots, P_N)$ expressed in some common numéraire unit, say US dollar. An asset related to a dividend paying stock is seen as a fund with the dividends re-invested. All assets are assumed to be positive semi-martingales living on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}(t) : t \geq 0\}$ is a right-continuous increasing family of σ-algebras, such that $\mathcal{F}(0)$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. As usual, we think of the filtration $\mathbb{F}$ as the carrier of information.

Uncertainty is introduced in two different ways: both continuously and through discrete events. More precisely, we assume that the filtered probability space is rich enough to carry a standard Brownian motion W, of dimension M, and an independent marked point process with measure p. The so-called mark space Υ represents the set of all types of discrete events and is, for simplicity, often assumed to be some Borel-measurable subset of $\mathbb{R}$ (typically an interval or a finite subset). Hence, informally, we can regard p as the measure of a multivariate point process (one point process for each $\upsilon \in \Upsilon$). We also assume that the predictable compensator of p admits a positive, time-varying, intensity measure λ, such that $\lambda(\Upsilon, t)$ is a.s. finite, for every $t \geq 0$. Finally, we let $\tilde{p}$ denote the measure of the corresponding compensated marked point process, and note that

$$\tilde{N}(t) = \int_0^t \int_\Upsilon \tilde{p}(d\upsilon, ds) = \int_0^t \int_\Upsilon (p(d\upsilon, ds) - \lambda(d\upsilon, s)ds), \tag{1}$$

is a pure-jump (local) martingale. As an example, we may think of a compensated Poisson process with constant intensity. In this case, the mark space contains just a single point, $\Upsilon = \{\upsilon_1\}$, at which the intensity measure has a constant point mass λ.

We proceed to build a model for our primary assets. First, let P_0 be the numéraire asset of the economy, describing how the value of the numéraire unit changes over time. For this asset, we require $P_0 > 0$ a.s. after which we introduce the relative asset prices $P_{0|n} = P_n/P_0$ according to

$$\frac{dP_{0|n}(t)}{P_{0|n}(t-)} = b_{0|n}(t)dt + \sum_{m=1}^{M} \sigma_{0|n,m}(t)dW^m(t) + \int_\Upsilon \alpha_{0|n}(\upsilon, t)\tilde{p}(d\upsilon, dt). \tag{2}$$

Here, the rate of excess return b_0 and the continuous volatility σ_0 are adapted processes, while the relative jump size α_0 must be a predictable process. Informally, we note that if there is a jump at time τ_υ, for the discrete event $\upsilon \in \Upsilon$, then

$$\Delta P_{0|n}(\tau_\upsilon) \triangleq P_{0|n}(\tau_\upsilon) - P_{0|n}(\tau_\upsilon -) = P_{0|n}(\tau_\upsilon -)\alpha_{0|n}(\upsilon, \tau_\upsilon). \tag{3}$$

In order to guarantee that a unique strong solution to Eq. (2) exists, we first impose the mild regularity condition

$$\int_0^T \left(\|b_0(t)\|_{\mathbb{R}^N} + \sum_{n=1}^{N}\sum_{m=1}^{M} \sigma^2_{0|n,m}(t) + \sum_{n=1}^{N} \int_\Upsilon \alpha^2_{0|n}(\upsilon, t)\lambda(d\upsilon, t) \right) dt < \infty,$$

almost surely, for every time horizon $[0, T]$, see Protter (2005) for further details. We also assume that $\alpha_{0|n} \geq -1$, for all $n \in \{1, \ldots, N\}$, to ensure that the solution is positive.

Remark 1 In most situations, the numéraire asset is assumed to be locally free and of the form

$$dP_0(t) = r(t)P_0(t)dt,$$

where the adaptive process r can be thought of as the interest rate in a savings account. Hence, under this assumption we can easily imply the dynamics

$$\frac{dP_n(t)}{P_n(t-)} = \frac{dP_{0|n}(t)}{P_{0|n}(t-)} + r(t)dt,$$

of the individual assets, which means that b_0 is the rate of return in excess of the interest rate.

An investor can trade in the assets, and throughout this paper we assume that there are no transaction fees, that short-selling is allowed, that trading takes place continuously in time, and that trading activity does not impact the asset prices. We define a trading strategy as a predictable vector process $w = (w^1, \dots, w^N)'$, representing the proportion of wealth invested in each asset, and let X_w denote the corresponding wealth process. We also set $X_{0|w} = X_w/P_0$. In this setup, the self-financing condition, see Geman et al. (1995), reads

$$\frac{dX_{0|w}(t)}{X_{0|w}(t-)} = \sum_{n=1}^{N} w^n(t) \frac{dP_{0|n}(t)}{P_{0|n}(t-)}, \tag{4}$$

It follows that the rate of excess return for the trading strategy w can undoubtedly be expressed as

$$b_{0|w}(t) = \sum_{n=1}^{N} w^n(t) b_{0|n}(t), \tag{5}$$

such that $b_{0|e_n} = b_{0|n}$, for the trading strategy $e_n = (0, \dots, 0, 1, 0, \dots, 0)'$ being the n'th coordinate vector corresponding to the investable assets. It is less clear how to associate a *total* instantaneous volatility with the wealth process. In order to explain the problem, we start by introducing the (local) martingale

$$Y_w(t) = \int_0^t w^n(s)\sigma_{0|n,m}(s)dW^m(s) + \int_0^t \int_\Upsilon w^n(s)\alpha_{0|n}(\upsilon, s)\tilde{p}(d\upsilon, ds),$$

using Einstein summation for repeated indices for notational simplicity, such that

$$dX_{0|w}(t) = X_{0|w}(t)b_{0|w}(t)dt + X_{0|w}(t-)dY_w(t). \tag{6}$$

Following Jacod and Shiryaev (2003), we then compute both the quadratic variation (bracket) process and the conditional quadratic variation (angle) process. The dynamics of these processes equals

$$d[Y_w, Y_w](t) = w^i(t)V^c_{0|i,j}(t)w^j(t)dt + \int_\Upsilon \left(w^i(t)\alpha_{0|i}(\upsilon, t)\right)^2 p(d\upsilon, dt),$$
$$d\langle Y_w, Y_w\rangle(t) = w^i(t)V^c_{0|i,j}(t)w^j(t)dt + \int_\Upsilon \left(w^i(t)\alpha_{0|i}(\upsilon, t)\right)^2 \lambda(d\upsilon, t)dt,$$

where

$$V^c_{0|i,j}(t) = \sum_{m=1}^{M} \sigma_{0|i,m}(t)\sigma_{0|j,m}(t), \tag{7}$$

denotes the instantaneous asset-asset covariance process associated with the continuous part of the martingale Y_w. One also sees that the contribution to the quadratic variation process, originating from the pure-jump martingale term of Y_w, equals $\sum_{s\le t}(\Delta Y_w(s))^2$ and that $[Y_w, Y_w] - \langle Y_w, Y_w\rangle$ is a pure-jump (local) martingale driven by the compensated marked point process. Since we want the *total* volatility to account for the discrete jumps in a smooth way (over time), we henceforth use the conditional quadratic covariation process; see also Björk and Slinko (2006); Christensen and Platen (2007). We now define

$$V_{0|v,w}(t) = \frac{d}{dt}\langle Y_v, Y_w\rangle(t), \tag{8}$$

$$\rho_{0|v,w}(t) = \frac{V_{0|v,w}(t)}{\sigma_{0|v}(t)\sigma_{0|w}(t)}, \quad \sigma^2_{0|w}(t) = V_{0|w,w}(t), \tag{9}$$

and observe that the instantaneous asset-asset covariance process can be decomposed into two distinct terms

$$V_{0|v,w}(t) = v^i(t)V_{0|i,j}(t)w^j(t), \quad V_{0|i,j}(t) = V^c_{0|i,j}(t) + V^d_{0|i,j}(t), \tag{10}$$

with V^c_0 being given by Eq. (7) and

$$V^d_{0|i,j}(t) = \int_\Upsilon \alpha_{0|i}(\upsilon, t)\alpha_{0|j}(\upsilon, t)\lambda(d\upsilon, t). \tag{11}$$

Remark 2 If the asset-asset covariance process V_0 is positive definite almost everywhere on $\mathbb{R}_+ \times \Omega$, it generates an inner product of the form $(v, w)_{V_0} = v^i V_{0|i,j} w^j$. Note that if both V^c_0 and V^d_0 are positive definite a.e. on $\mathbb{R}_+ \times \Omega$, then so is V_0. Henceforth, we always assume this to be the case, which implies that the inverse of V_0 exists and that V_0^{-1} is also a.e. positive definite. We also write $\|w\|^2_A = (w, w)_A$ whenever the inner product is well defined for some matrix process A.

3 Optimal portfolio strategies

In this Section, we take a closer look at two prominent trading strategies that deviate from the expected utility approach of von Neumann and Morgenstern (1947) and, consequently, are somewhat different in spirit from the mean-variance approach of Markowitz (1952). In the presence of a risk-free asset, what characterizes the optimal allocations in the (single period) mean-variance approach is that they all have maximal Sharpe ratios in the sense of Sharpe (1966, 1994). These strategies are said to lie on the *efficient frontier*. The analogous trading strategies in a continuous-time framework are typically referred to as maximal Sharpe strategies; see, for example, Christensen and Platen (2007) and the references therein. In the sequel, we first consider an investor who maximizes the instantaneous Sharpe ratio and thereafter we consider a so-called Kelly trader. It is known that in the absence of jumps in the primary assets, these two approaches lead to the same portfolio allocations; see, for example, Bermin and

Holm (2021, 2023). Hence, the allocations are characterized by having a maximal instantaneous Sharpe ratio in the sense of Nielsen and Vassalou (2004). Below, we show that with jumps present this property no longer holds. We also include some general remarks regarding the connection to optimal expected utility strategies.

3.1 Sharpe maximizing strategies

We extend the instantaneous Sharpe ratio, as introduced in Nielsen and Vassalou (2004), to our jump-diffusion setting (similar to Björk and Slinko (2006) and Christensen and Platen (2007)) according to

$$s_{0|w}(t) = \frac{b_{0|w}(t)}{\sigma_{0|w}(t)}. \tag{12}$$

As in Bermin and Holm (2021, 2023), it now follows from Eqs. (4) and (10), Remark 2, and Cauchy–Schwartz's inequality that

$$s^2_{0|w}(t) = \frac{\big(w(t), \hat{w}(t)\big)^2_{V_0(t)}}{(w(t), w(t))_{V_0(t)}} \leq \big(\hat{w}(t), \hat{w}(t)\big)_{V_0(t)}, \quad \hat{w}(t) = V_0^{-1}(t) b_0(t), \tag{13}$$

with equality if and only if w and $\hat{w}$ are collinear; that is, if $w = k\hat{w}$ for some predictable real-valued process k. Hence, the maximal instantaneous Sharpe ratio satisfies

$$|s_{0|k\hat{w}}(t)| = \|b_0(t)\|_{V_0^{-1}(t)}, \tag{14}$$

independently of k. We call the set of strategies collinear with $\hat{w} = V_0^{-1} b_0$ for the *local efficient frontier*. Furthermore, the local characteristics of a Sharpe maximizing strategy can be summarized as

$$b_{0|k\hat{w}}(t) = k(t) s^2_{0|\hat{w}}(t), \quad \sigma^2_{0|k\hat{w}}(t) = k^2(t) s^2_{0|\hat{w}}(t). \tag{15}$$

In order to describe the rate of excess logarithmic return, we apply the generalized Itô formula, see Protter (2005). For notational simplicity we, again, use Einstein summation over repeated indices

$$\begin{aligned} d \ln X_{0|w}(t) = w^i(t) \left(b_{0|i}(t) - \frac{1}{2} V^c_{0|i,j}(t) w^j(t) - \int_\Upsilon \alpha_{0|i}(\upsilon, t) \lambda(d\upsilon, t) \right) dt \\ + w^n(t) \sigma_{0|n,m}(t) dW^m(t) + \int_\Upsilon \Delta \ln X_{0|w}(t) p(d\upsilon, dt). \end{aligned}$$

For those unfamiliar with jumps in the Itô formula, we quickly mention that one can essentially proceed as if no jumps exist (i.e. ignoring the measure p of the marked point process) and thereafter correct for the jumps. We summarize the result in the following.

Lemma 1 *Let the dynamics of the wealth process be given by*

$$\frac{dX_{0|w}(t)}{X_{0|w}(t-)} = b_{0|w}(t)dt + w^n(t)\sigma_{0|n,m}(t)dW^m(t) + \int_\Upsilon w^n(t)\alpha_{0|n}(\upsilon,t)\tilde{p}(d\upsilon,dt).$$

Then

$$d\ln X_{0|w}(t) = \left(b_{0|w}(t) - \frac{1}{2}\sigma^2_{0|w}(t) + \int_\Upsilon h\left(w^n(t)\alpha_{0|n}(\upsilon,t)\right)\lambda(d\upsilon,t)\right)dt$$
$$+ w^n(t)\sigma_{0|n,m}(t)dW^m(t) + \int_\Upsilon \ln\left(1 + w^n(t)\alpha_{0|n}(\upsilon,t)\right)\tilde{p}(d\upsilon,dt),$$

where

$$h(x) = \ln(1+x) - x + \frac{1}{2}x^2 \approx \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \ldots$$

Proof If there is a jump at time τ_υ, for the discrete event $\upsilon \in \Upsilon$, then

$$\Delta X_{0|w}(\tau_\upsilon) = X_{0|w}(\tau_\upsilon) - X_{0|w}(\tau_\upsilon-) = X_{0|w}(\tau_\upsilon-)w^n(\tau_\upsilon)\alpha_{0|n}(\upsilon,\tau_\upsilon),$$

from which we obtain

$$\Delta \ln X_{0|w}(\tau_\upsilon) = \ln\left(1 + w^n(\tau_\upsilon)\alpha_{0|n}(\upsilon,\tau_\upsilon)\right).$$

The proof concludes by replacing V_0^c with $V_0 - V_0^d$, as in Eq. (10), and expressing the result as a semi-martingale. □

To prevent the wealth process from being over-exposed to an individual asset or from becoming negative, it must also hold that $w^n \cdot \alpha_{0|n} \geq -1$, for $1 \leq n \leq N$, and $\sum_{n=1}^N w^n\alpha_{0|n} \geq -1$ a.e. in $\mathbb{R}_+ \times \Upsilon \times \Omega$. We let $\mathcal{A}$ denote the class of all trading strategies such that $X_{0|w} \geq 0$ almost surely. In light of Lemma 1, we now define the rate of excess logarithmic return by

$$\mu_{0|w}(t) = b_{0|w}(t) - \frac{1}{2}\sigma^2_{0|w}(t) + \int_\Upsilon h\left(w^n(t)\alpha_{0|n}(\upsilon,t)\right)\lambda(d\upsilon,t), \tag{16}$$

and observe, using Eq. (15), that for any Sharpe maximizing strategy, we have

$$\mu_{0|k\hat{w}}(t) = \frac{1}{2}k(t)\left(2 - k(t)\right)s^2_{0|\hat{w}}(t) + \int_\Upsilon h\left(k(t)\hat{w}^n(t)\alpha_{0|n}(\upsilon,t)\right)\lambda(d\upsilon,t). \tag{17}$$

In order to quantify the impact of jumps, we plot, in Fig. 1, the increasing function h. We see that if the magnitude of the relative jump size is less than 25%, there is very little impact on the rate of excess logarithmic return: $h(0.25) = 0.4\%$ and $h(-0.25) = -0.6\%$. To put these numbers in perspective, it is illustrative to recall that the largest daily percentage loss for the S&P500 index, over the past 100 years,

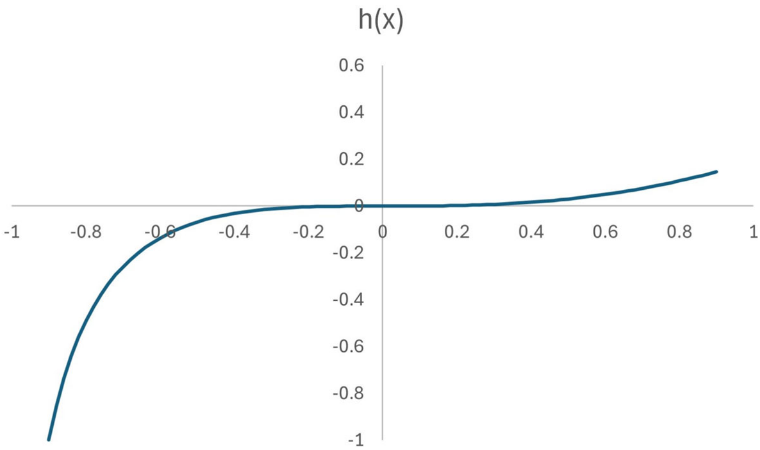

Fig. 1 This figure shows the function h over the interval $[-0.9, 0.9]$ of relative jump sizes for the wealth process. Note that the function is roughly flat for relative jumps with magnitude less than 25%

occurred in 1987 and measured approximately -21%. For jump sizes with higher magnitude, either corresponding to rather risky assets or to leveraged positions in moderately risky assets, we further notice that the contribution is asymmetric and that negative large jumps have greater impact than positive large jumps of the same size. The reason is simply that bankruptcy is an absorbing state.

In order to evaluate the maximal rate of excess logarithmic return that a maximal Sharpe investor can achieve, we set $k_* = \arg\max_k \mu_{0|k\hat{w}}$ and refer to the following result.

Theorem 2 *Let k_* be the solution to the non-linear integral equation*

$$k_*(t) - \frac{1}{s^2_{0|\hat{w}}(t)} \int_{\Upsilon} h'\left(k_*(t)L(\upsilon, t)\right) L(\upsilon, t)\lambda(d\upsilon, t) = 1,$$

where

$$L(\upsilon, t) = \left(b_0(t), \alpha_0(\upsilon, t)\right)_{V_0^{-1}(t)}.$$

Then, if $k_*L \geq -1$ a.s., the growth optimal maximal Sharpe strategy $w = k_*\hat{w}$ yields the highest rate of excess logarithmic return $\mu_{0|w}$ among all maximal Sharpe strategies.

Proof The proof follows from the first-order condition associated with Eq. (17) □

From the above result, we notice that in the absence of jumps (when the function h vanishes) we immediately get $k_* = 1$. However, in general, the optimal leverage term must be computed numerically. We end our discussion about maximal Sharpe strategies by providing a simple example when the leverage term k_* can, in fact, be calculated analytically.

Example 1 In the special case where the mark space contains just a single point, $\Upsilon = \{\upsilon_1\}$, at which the intensity measure has a point mass $\lambda(\cdot) = \lambda(\Upsilon, \cdot)$, the maximal Sharpe strategy $k_* \hat{w}$ is characterized by

$$k_*(t) = \frac{L(t) - 1 + \sqrt{(L(t)+1)^2 - 4\lambda(t)L^3(t)/B(t)}}{2L(t)\left(1 - \lambda(t)L^2(t)/B(t)\right)},$$

where we have set

$$L(t) = L(\upsilon_1, t), \quad B(t) = \|b_0(t)\|^2_{V_0^{-1}(t)} = s^2_{0|\hat{w}}(t).$$

The calculations follow from the identity $(1+x)h'(x) = x^2$.

3.2 Kelly strategies

In Kelly (1956) the long-term performance of certain trading strategies was considered in conjunction with the law of large numbers. Nowadays, we say that a trading strategy w is *Kelly-admissible* if $w \in \mathcal{A}$ (i.e. if $X_{0|w} \geq 0$) and

$$\limsup_{T\to\infty} \frac{1}{T} \ln \frac{X_{0|w}(T)}{X_{0|w}(0)} = \limsup_{T\to\infty} \frac{1}{T} \int_0^T \mu_{0|w}(t)dt, \quad a.s. \tag{18}$$

We also let $\mathcal{A}_*$ denote the class of all such trading strategies. Hence, by applying the strategy $w_* = \arg\max_{w\in\mathcal{A}_*} \mu_{0|w}$, Kelly drew the conclusion that "our gambler's capital will surpass, with probability one, that of any other gambler apportioning his money differently". Although this statement is certainly true for trading strategies in $\mathcal{A}_*$, it might not extend to the larger class $\mathcal{A}$; a technical topic that we leave aside for now[1]. Instead, we highlight the observation that, in the absence of asset price jumps, Eq. (16) yields

$$w_*(t) = \arg\max_{w(t)}\{b_{0|w}(t) - \frac{1}{2}\sigma^2_{0|w}(t)\} = V_0^{-1}(t)b_0(t) = \hat{w}(t), \tag{19}$$

and therefore the instantaneous Sharpe ratio of w_* is maximal with

$$\mu_{0|w_*}(t) = \frac{1}{2}s^2_{0|\hat{w}}(t). \tag{20}$$

Hence, using a very different approach, Kelly essentially derived the same asset allocations (that is, maximal Sharpe ratio) as did Markowitz in his celebrated mean-variance framework. The main difference being that Kelly also provided a long-term estimate, Eq. (18), for the amount of money that could be earned (with probability one) by

[1] For those interested in precise conditions for when the strong law of large numbers applies to a local martingale, we refer to Liptser and Shiryayev (1989), Sect. 2.6.

dynamically trading the position. It is also important to understand that Kelly applied the logarithmic function to the terminal portfolio only as a means to subsequently use the law of large numbers. We now proceed with jumps present. Below, we show that, in this case, the growth optimal Kelly strategy w_* is typically no longer collinear with the maximal Sharpe strategy $\hat{w}$.

Theorem 3 *Let ϕ be the solution to the non-linear integral equation*

$$\phi(u,t) - \int_{\Upsilon} h'\left(\phi(\upsilon,t)\right) Q(u,\upsilon,t)\lambda(d\upsilon,t) = L(u,t), \quad u \in \Upsilon,$$

where L and the symmetric kernel Q equals

$$L(u,t) = (b_0(u,t), \alpha_0(u,t))_{V_0^{-1}(t)}, \quad Q(u,\upsilon,t) = (\alpha_0(u,t), \alpha_0(\upsilon,t))_{V_0^{-1}(t)}.$$

Then, if $\phi \geq -1$ a.s., the growth optimal Kelly strategy $w_* = \arg\max_w \mu_{0|w}$ equals

$$w_*(t) = \hat{w}(t) + \int_{\Upsilon} h'\left(\phi(\upsilon,t)\right) V_0^{-1}(t)\alpha_0(\upsilon,t)\lambda(d\upsilon,t).$$

Proof The proof is a direct consequence of the first-order condition, associated with Eq. (16), which reads

$$w_*(t) = \hat{w}(t) + \int_{\Upsilon} h'\left(w_*^n(t)\alpha_{0|n}(\upsilon,t)\right) V_0^{-1}(t)\alpha_0(\upsilon,t)\lambda(d\upsilon,t).$$

Multiplying both sides by α_0 concludes. □

Although the integral equation, in ϕ, shows similarities with a Fredholm equation of the second kind, the nonlinearity of h' forces us to apply numerical methods. In Sect. 4 we show that, in many cases, such numerical schemes are highly efficient and simple to apply. However, before going there, we provide an example of when the integral equation can be solved analytically.

Example 2 We consider again the special case where the mark space contains just a single point, $\Upsilon = \{\upsilon_1\}$, at which the intensity measure has a point mass $\lambda(\cdot) = \lambda(\Upsilon, \cdot)$. Then, from the identity $(1+x)h'(x) = x^2$, we calculate

$$\phi(\upsilon_1,t) = \frac{L(t) - 1 + \sqrt{(L(t)+1)^2 - 4\lambda(t)L(t)Q(t)}}{2\left(1 - \lambda(t)Q(t)\right)},$$
$$L(t) = L\left(\upsilon_1,t\right), \quad Q(t) = Q\left(\upsilon_1,\upsilon_1,t\right).$$

By comparing with Example 1, it follows that $w_*^n\alpha_{0|n} = k_*\hat{w}^n\alpha_{0|n}$ if and only if $L^2 = BQ$, where $B = \|b_0\|^2_{V_0^{-1}}$, and from Cauchy-Schwartz we know that the latter expression is true if and only if b_0 and α_0 are linearly dependent.

Kelly strategies are defined as strategies collinear to the growth optimal Kelly strategy; that is, of the form $w = kw_*$ for some predictable real-valued process k. Hence, we can regard these strategies as a slight generalization of the fractional Kelly strategies in MacLean et al. (1992), and we note that they represent the *local Kelly frontier*, consisting of all strategies that have the same instantaneous Sharpe ratio as the growth optimal Kelly strategy.

Corollary 4 *Let k be a predictable real-valued process. Then*

$$b_{0|kw_*}(t) = k(t)b_{0|w_*}(t), \quad \sigma^2_{0|kw_*}(t) = k^2(t)\sigma^2_{0|w_*}(t).$$

With the notations of Theorem 3, we further have

$$\begin{aligned} b_{0|w_*}(t) &= b_{0|\hat{w}}(t) + \int_\Upsilon h'\left(\phi(\upsilon, t)\right) L(\upsilon, t)\lambda(d\upsilon, t), \\ \sigma^2_{0|w_*}(t) &= \sigma^2_{0|\hat{w}}(t) + \int_\Upsilon h'\left(\phi(\upsilon, t)\right)\left(L(\upsilon, t) + \phi(\upsilon, t)\right)\lambda(d\upsilon, t), \end{aligned}$$

such that the local rate of logarithmic return equals

$$\mu_{0|kw_*}(t) = b_{0|kw_*}(t) - \frac{1}{2}\sigma^2_{0|kw_*}(t) + \int_\Upsilon h\left(k\phi(\upsilon, t)\right)\lambda(d\upsilon, t).$$

Proof The first part of the proof follows from the general expressions $b_{0|w} = \left(w, \hat{w}\right)_{V_0}$, where $\hat{w} = V_0^{-1}b_0$, and $\sigma^2_{0|w} = (w, w)_{V_0}$. Using Theorem 3, the expression for $b_{0|w_*}$ immediately follows, while we calculate

$$\begin{aligned} \sigma^2_{0|w_*}(t) = \sigma^2_{0|\hat{w}}(t) &+ 2\int_\Upsilon h'\left(\phi(\upsilon, t)\right) L(\upsilon, t)\lambda(d\upsilon, t) \\ &+ \int_\Upsilon\int_\Upsilon h'\left(\phi(u, t)\right) Q(u, \upsilon, t)h'\left(\phi(\upsilon, t)\right)\lambda(du, t)\lambda(d\upsilon, t). \end{aligned}$$

Using Theorem 3 and the equation determining ϕ, we further see that

$$\begin{aligned} &\int_\Upsilon\int_\Upsilon h'\left(\phi(u, t)\right) Q(u, \upsilon, t)h'\left(\phi(\upsilon, t)\right)\lambda(du, t)\lambda(d\upsilon, t) \\ &\quad = \int_\Upsilon h'\left(\phi(\upsilon, t)\right)\phi(\upsilon, t)\lambda(d\upsilon, t) - \int_\Upsilon h'\left(\phi(\upsilon, t)\right) L(\upsilon, t)\lambda(d\upsilon, t). \end{aligned}$$

The final part is a direct consequence of Eq. (16). □

We end the characterization of Kelly strategies by comparing the growth optimal Kelly strategy w_* with the growth optimal maximal Sharpe strategy $k_*\hat{w}$.

Proposition 5 *With the notations of Theorem 3, we have*

$$\begin{aligned}
b_{0|w_*}(t) - b_{0|k_*\hat{w}}(t) &= \int_\Upsilon \{h'(\phi(\upsilon,t)) - h'(k_*(t)L(\upsilon,t))\}L(\upsilon,t)\lambda(d\upsilon,t),\\
\sigma^2_{0|w_*}(t) - \sigma^2_{0|k_*\hat{w}}(t) &= \int_\Upsilon \{h'(\phi(\upsilon,t)) - h'(k_*(t)L(\upsilon,t))\}L(\upsilon,t)\lambda(d\upsilon,t)\\
&\quad + \int_\Upsilon \{h'(\phi(\upsilon,t))\,\phi(\upsilon,t) - h'(k_*(t)L(\upsilon,t))\,k_*(t)L(\upsilon,t)\}\lambda(d\upsilon,t),\\
\mu_{0|w_*}(t) - \mu_{0|k_*\hat{w}}(t) &= \frac{1}{2}\int_\Upsilon \{h'(\phi(\upsilon,t)) - h'(k_*(t)L(\upsilon,t))\}L(\upsilon,t)\lambda(d\upsilon,t)\\
&\quad - \frac{1}{2}\int_\Upsilon \{h'(\phi(\upsilon,t))\,\phi(\upsilon,t) - h'(k_*(t)L(\upsilon,t))\,k_*(t)L(\upsilon,t)\}\lambda(d\upsilon,t)\\
&\quad + \int_\Upsilon \{h(\phi(\upsilon,t)) - h(k_*(t)L(\upsilon,t))\}\lambda(d\upsilon,t).
\end{aligned}$$

Hence, a sufficient condition for w_* and $k_*\hat{w}$ to have the same local dynamics is that $\phi(\upsilon,t) = k_*(t)L(\upsilon,t)$ for all $\upsilon \in \Upsilon$. The necessary condition is that the three distinct integrals vanish.

Proof Using Theorem 2, it follows that

$$b_{0|k_*\hat{w}}(t) = k_*(t)b_{0|\hat{w}}(t) = b_{0|\hat{w}}(t) + \int_\Upsilon h(k_*(t)L(\upsilon,t))\,L(\upsilon,t)\lambda(d\upsilon,t).$$

Moreover, since $b_{0|\hat{w}} = \sigma^2_{0|\hat{w}} = s^2_{0|\hat{w}}$, we also have

$$\begin{aligned}
\sigma^2_{0|k_*\hat{w}}(t) = k_*^2(t)\sigma^2_{0|\hat{w}}(t) &= k_*(t)\sigma^2_{0|\hat{w}}(t) + \int_\Upsilon h(k_*(t)L(\upsilon,t))\,k_*(t)L(\upsilon,t)\lambda(d\upsilon,t)\\
&= \sigma^2_{0|\hat{w}}(t) + (1+k_*(t))\int_\Upsilon h(k_*(t)L(\upsilon,t))\,L(\upsilon,t)\lambda(d\upsilon,t),
\end{aligned}$$

from which the proof follows by Corollary 4. □

3.3 Optimal expected utility strategies

Here, we briefly comment on the connection to optimal expected utility strategies, defined as $\bar{w} = \arg\max_{w\in\mathcal{A}} \mathbb{E}\left[U\left(X_{0|w}(T)\right)\right]$, where U denotes a utility function. However, based on the results in Hakansson (1971) and Elton and Gruber (1974), we limit the scope to the power utility function

$$U(x) = \frac{x^{1-p}-1}{1-p}, \quad p > 0,$$

where p represents the relative risk aversion. In the limit as $p \to 1$, it follows that $U(x) = \ln x$. This case was carefully studied in Goll and Kallsen (2003), where it was shown that the growth optimal Kelly strategy w_* equally solves the optimal expected utility problem, within a general jump-diffusion setting, for any horizon T. For other values of p the situation is more complex, even in the absence of jumps. By analyzing the results in, for example, Björk et al. (2010); Callegaro et al. (2006), one finds that the optimal expected utility strategy is of the form $\bar{w} = \frac{1}{p} w_* + \tilde{w}$, where the strategy $\tilde{w}$ represents an intertemporal hedge to the randomness associated with (in particular) the instantaneous Sharpe ratio. Hence, with deterministic model parameters, or when the noise of the factors driving the model parameters is uncorrelated with the noise driving the asset prices, as in Davis and Lleo (2013), this term vanishes, and we recover the functional form of the Kelly strategies. From an economic point of view, however, it is somewhat unsettling that utility investors only hedge the parameter risk when $p \neq 1$ but not when $p = 1$. In practice, the chosen method depends on whether an investor believes that maximizing utility is a good way to balance return versus risk, or whether other risk/performance criteria are more important.

4 Two case studies

In this Section, we provide two artificial but illustrative case studies that highlight the impact of jumps on asset allocation. In the first example, we focus on the fear of a financial crash, while in the second example, we focus on asset picking skills. The goal of this Section is to emphasize that a Kelly strategy is not necessarily a maximal Sharpe strategy in the presence of jumps.

In order to explain the model we attempt to analyze, we have chosen to keep it as simple as possible. This means that we have removed any randomness and time dependency in the model parameters. We have also chosen to consider trading in two assets only, and we assume that the mark space contains three points, $\Upsilon = \{\upsilon_1, \upsilon_2, \upsilon_3\}$. At each of these points, the intensity measure has a constant point mass $(\lambda_1, \lambda_2, \lambda_3)$. For each discrete event in Υ we further specify a constant relative jump size $(\alpha^v_{0|1}, \alpha^v_{0|2})$, $v \in \{1, 2, 3\}$. This concludes the jump specification. The diffusion specification is expressed using the volatility of the individual assets $(\sigma_{0|1}, \sigma_{0|2})$ together with the Brownian correlation $\rho_{0|1,2}$. Finally, we let $(b_{0|1}, b_{0|2})$ denote the rate of individual excess asset returns.

The Kelly strategy is constructed from Theorem 3 using Newton's method. That is, for each discrete event, we find the roots (ϕ_1, ϕ_2, ϕ_3) to

$$F_u(\phi_1, \phi_2, \phi_3) = \phi_u - \sum_{\upsilon=1}^{3} h'(\phi_\upsilon)\, Q_{u,\upsilon}\lambda_\upsilon - L_u, \quad u \in \{1, 2, 3\},$$

by applying the iterative procedure $\phi \mapsto \phi - J^{-1}F$, with the Jacobian defined such that $J_{u,\upsilon} = \partial F_u / \partial \phi_\upsilon = \mathbf{1}\{u = \upsilon\} - h''(\phi_\upsilon)\, Q_{u,\upsilon}\lambda_\upsilon$. Typically, convergence occurs within five iterations from the initial guess $\phi_u = \hat{w}^n \alpha^u_{0|n}$. We also apply the same method to Theorem 2 when searching for k_*, but since this is standard, we omit the details.

Table 1 The model states that once every 10 years (on average) the market will crash: asset 1 will loose 20% of its value, while asset 2 will loose 45%. Additionally, once every 5 years (on average) the market will correct itself: asset 1 will adjust downwards by 10%, while asset 2 drops by 30%. On the positive side, once every 2 years (on average) the assets will jump upwards with 4% and 6%, respectively

Event υ	$\alpha^{\upsilon}_{0\|1}$	$\alpha^{\upsilon}_{0\|2}$	λ_{υ}
1	−20%	−45%	0.1
2	−10%	−30%	0.2
3	4%	6%	0.5

Because this Section is about how to quantify an investor's perception of jumps, we keep the diffusion part fixed. The parameters we have chosen are: $\sigma_{0|1} = 20\%$, $\sigma_{0|2} = 30\%$, and $\rho_{0|1,2} = 60\%$. This yields the continuous instantaneous covariance matrix

$$V_0^c = \begin{pmatrix} 0.040 & 0.036 \\ 0.036 & 0.090 \end{pmatrix}. \tag{21}$$

In order to explain why the second asset is assumed to be more volatile, we consider two distinct cases.

4.1 Fearing a crash

Consider an investor who is concerned about the next financial crash. His perception of the two assets in which he trades is described in Table 1.

Clearly, he fears a gloomy future and wants to allocate his assets accordingly. The perceived jumps add volatility to the assets, as described in Eqs. (10) and (11), according to

$$V_0^d = \begin{pmatrix} 0.007 & 0.016 \\ 0.016 & 0.040 \end{pmatrix}. \tag{22}$$

It is interesting to note that the jump correlation, that is, the correlation implied from V_0^d is approximately 98%. Hence, the investor could basically have modeled the discrete events with a single mark[2]. With that being said, there is evidence; see, for example, Sandoval and Franca (2012) that large financial crashes tend to be highly correlated among assets. Yet, the *total* correlation between the two assets amounts to 67%.

We now set the rate of excess returns to: $b_{0|1} = 12\%$ and $b_{0|2} = 20\%$, such that they have the same Sharpe ratios (0.555 to be precise). This yields the asset allocations

$$\hat{w} = \begin{pmatrix} 1.537 \\ 0.921 \end{pmatrix}, \quad k_* \hat{w} = \begin{pmatrix} 1.233 \\ 0.739 \end{pmatrix}, \quad w_* = \begin{pmatrix} 1.508 \\ 0.597 \end{pmatrix}. \tag{23}$$

[2] Note that the jump correlation, $V^d_{0|1,2}/\sqrt{V^d_{0|1,1} V^d_{0|2,2}}$, equals 100% if the mark space Υ contains only a single point.

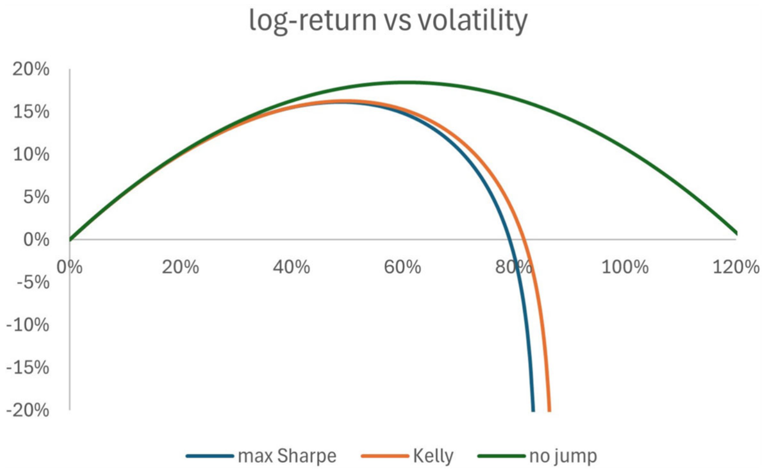

Fig. 2 This figure shows the rate of excess logarithmic return $\mu_{0|w}$ as a function of the volatility $\sigma_{0|w}$ for the strategies: $w = k\hat{w}$ (maximal Sharpe) and $w = kw_*$ (Kelly), with k used to generate various combinations. The jump specification is given by Table 1 (*fearing a crash*). We also compare to a situation with no jumps, that is where the covariance matrix V_0 is fully allocated to the continuous part V_0^c

Table 2 This table shows the local characteristics of the growth optimal maximal Sharpe and Kelly strategies. Note that the maximal Sharpe strategy has a higher instantaneous Sharpe ratio, while the Kelly strategy has a higher rate of excess logarithmic return

Strategy w	$b_{0\|w}$	$\sigma_{0\|w}$	$s_{0\|w}$	$\mu_{0\|w}$
$k_*\hat{w}$	29.6%	48.7%	0.607	16.2%
w_*	30.0%	49.7%	0.605	16.3%

Hence, the maximal Sharpe strategy allocates 1.7 times more wealth to asset 1, while the Kelly strategy allocates 2.5 times more wealth to asset 1. In other words, a Kelly trader fears market crashes more than a maximal Sharpe investor. For further details on these strategies, see Table 2. Note that the Kelly strategy has a slightly higher rate of excess logarithmic return but a slightly lower instantaneous Sharpe ratio than the growth optimal maximal Sharpe strategy.

Finally, we address the question of robustness. We evaluate the worst-case scenario for the relative jumps, $\min_\upsilon(w^1\alpha^\upsilon_{0|1} + w^2\alpha^\upsilon_{0|2})$, and find this expression to equal: -0.58 for $w = k_*\hat{w}$ and -0.57 for $w = w_*$. Hence, by leveraging these strategies, we expect the maximal Sharpe strategy to go bankrupt slightly before the Kelly strategy. Another way to highlight this behavior is to plot, as in Fig. 2, the rate of excess logarithmic return $\mu_{0|kw}$ against its volatility $\sigma_{0|kw}$, as k varies, for the strategies $w = \hat{w}$ and $w = w_*$. This point of view indicates that the qualitative differences between the two trading strategies are small and manifest only with high leverage. Another result that we have observed is that the outcome mainly depends on the worst-case event $\upsilon = 1$. That is, if the relative jump sizes $\alpha^\upsilon_{0|1}$ and $\alpha^\upsilon_{0|2}$ are set to zero, for $\upsilon = 2$ and 3, and we simply re-calibrate the rate of excess returns, $b_{0|1}$ and $b_{0|2}$, to match the original

Table 3 The model states that once every 10 years (on average) the investor's asset picking skills result in: asset 1 has a modest jump of 10%, while asset 2 makes a breakthrough and triple in price. Additionally, once every 5 years (on average) the market will correct itself: asset 1 will adjust downwards by 10%, while asset 2 drops by 30%. In the short run, once every 2 years (on average), the investor believes that asset 1 will remain neutral, while asset 2 will jump upwards with 10%

Event υ	$\alpha^{\upsilon}_{0\|1}$	$\alpha^{\upsilon}_{0\|2}$	λ_{υ}
1	10%	200%	0.1
2	−10%	−30%	0.2
3	0%	10%	0.5

instantaneous Sharpe ratios, $s_{0|1}$ and $s_{0|2}$, there is hardly any noticeable effect on any of the quantities reported.

4.2 Picking a winner

We now consider an investor, full of confidence in his ability to pick assets that may outperform the market in the near future.

Again, the perceived jumps, outlined in Table 3, add volatility to the assets according to

$$V_0^d = \begin{pmatrix} 0.003 & 0.026 \\ 0.026 & 0.423 \end{pmatrix}. \tag{24}$$

In this scenario, the jump correlation is approximately 73%, while the *total* correlation between the two assets equals 42%.

We now set the rate of excess returns to: $b_{0|1} = 12\%$ and $b_{0|2} = 40\%$, such that they have roughly the same Sharpe ratios (0.579 and 0.558, respectively, to be precise). This yields the asset allocations

$$\hat{w} = \begin{pmatrix} 2.018 \\ 0.536 \end{pmatrix}, \quad k_*\hat{w} = \begin{pmatrix} 2.433 \\ 0.646 \end{pmatrix}, \quad w_* = \begin{pmatrix} 0.811 \\ 1.551 \end{pmatrix}. \tag{25}$$

It is interesting to note that the maximal Sharpe strategy allocates 3.8 times more wealth to asset 1, while the Kelly strategy allocates 1.9 times more wealth to asset 2. In other words, a Kelly trader favors assets with huge upward potential much more than a maximal Sharpe investor. For further details on these strategies, see Table 4. Again, we note that the Kelly strategy has a higher rate of excess logarithmic return but a lower instantaneous Sharpe ratio than the growth optimal maximal Sharpe strategy.

Regarding robustness, we notice that $\min_{\upsilon}(w^1\alpha^{\upsilon}_{0|1} + w^2\alpha^{\upsilon}_{0|2})$ equals: -0.44 for $w = k_*\hat{w}$ and -0.55 for $w = w_*$. Hence, the Kelly strategy seems to be riskier than the maximal Sharpe strategy at first sight. However, when plotting the rate of excess logarithmic return against the volatility, the picture changes. From Fig. 3 it is clear that the Kelly strategy is the more robust, i.e. can be leveraged harder. A more thorough analysis shows that the outcome is mainly driven by the events $\upsilon = 1$ and 2. As in the

Table 4 This table shows the local characteristics of the growth optimal maximal Sharpe and Kelly strategies. Note that the maximal Sharpe strategy has a higher instantaneous Sharpe ratio, while the Kelly strategy has a higher rate of excess logarithmic return

Strategy w	$b_{0\|w}$	$\sigma_{0\|w}$	$s_{0\|w}$	$\mu_{0\|w}$
$k_*\hat{w}$	55.0%	81.5%	0.676	26.8%
w_*	71.8%	119.1%	0.603	32.2%

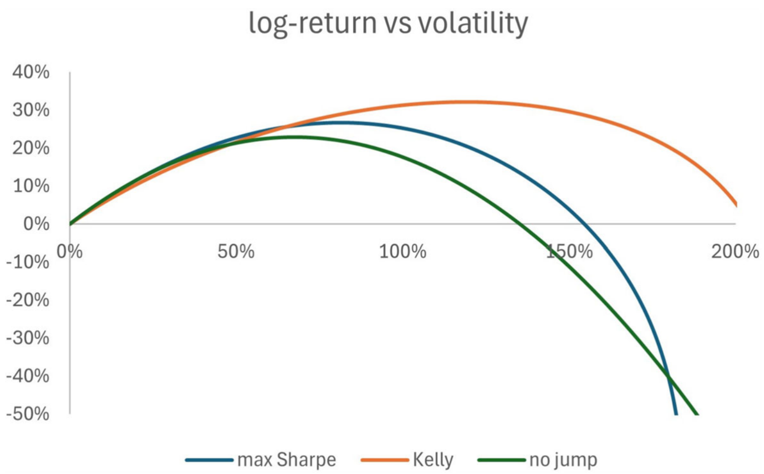

Fig. 3 This figure shows the rate of excess logarithmic return $\mu_{0|w}$ as a function of the volatility $\sigma_{0|w}$ for the strategies: $w = k\hat{w}$ (maximal Sharpe) and $w = kw_*$ (Kelly), with k used to generate various combinations. The jump specification is given by Table 3 (*picking a winner*). We also compare to a situation with no jumps, that is where the covariance matrix V_0 is fully allocated to the continuous part V_0^c

previous case study, we find that the two strategies respond similarly to the worst-case event $\upsilon = 2$. Hence, it is the best-case event $\upsilon = 1$ that accounts for the differences. For low and moderate levels of volatility, the Kelly strategy gives up a small amount of the Sharpe ratio and in return it outperforms for high levels of volatility.

5 Comments on random jump sizes

In this Section, we show how to deal with random jump sizes, drawing inspiration from Merton and his jump-diffusion model (Merton (1976)). We also show that the case studies in the previous Section fit with this new interpretation.

We proceed by setting the mark space to $\Upsilon = [-1, \infty)^N$. Next, define the intensity measure by

$$\lambda(d\upsilon, t) = \lambda(t) f_{\alpha_0}(\upsilon_1, \ldots, \upsilon_N, t) d\upsilon_1 \cdots d\upsilon_N = \lambda(t) f_{\alpha_0}(\upsilon, t) d^N \upsilon, \qquad (26)$$

where λ denotes the intensity of the driving point process p while f_{α_0} denotes the density of the jumps $\alpha_0 = (\alpha_{0|1}, \ldots, \alpha_{0|N})$.

Example 3 The jump specifications in Sect. 4 can equally be characterized by a generalized density function and a constant intensity measure as described below

$$f_{\alpha_0}(\upsilon_1, \upsilon_2) = \sum_{i=1}^{3} \delta_0(\upsilon_1 - \alpha^i_{0|1})\delta_0(\upsilon_2 - \alpha^i_{0|2})p_i,$$

where δ_0 denotes Dirac's delta function and

$$p_i = \frac{\lambda_i}{\lambda}, \quad \lambda = \sum_{i=1}^{3} \lambda_i.$$

The formula follows from identifying all jumps and multiplying with the corresponding probability.

In most cases, though, we can work with a reduced mark space. To illustrate, we set

$$f^{(i,j)}_{\alpha_0}(\upsilon_i, \upsilon_j, t) = \int_{[-1,\infty)^{N-2}} f_{\alpha_0}(\upsilon_1, \ldots, \upsilon_N, t) \prod_{n=1, n\notin\{i,j\}}^{N} d\upsilon_n. \tag{27}$$

Then, for instance, the instantaneous discrete covariance matrix in Eq. (11) takes the form

$$V^d_{0|i,j}(t) = \lambda(t) \int_{[-1,\infty)^2} \upsilon_i \upsilon_j f^{(i,j)}_{\alpha_0}(\upsilon_i, \upsilon_j, t) d\upsilon_i d\upsilon_j. \tag{28}$$

Let us now consider Merton's approach. First, we transform the random jump vector $\alpha_0 = (\alpha_{0|1}, \ldots, \alpha_{0|N})$ to $\beta_0 = (\beta_{0|1}, \ldots, \beta_{0|N})$, according to $\beta_{0|n} = \ln(1 + \alpha_{0|n})$. We then write

$$\mu_{0|w}(t) = b_{0|w}(t) - \frac{1}{2}\sigma^2_{0|w}(t) + \lambda(t) \int_{\Upsilon} h\left(\sum_{n=1}^{N} w^n \left(e^{\upsilon_n} - 1\right)\right) f_{\beta_0}(\upsilon, t) d^N\upsilon,$$

with $\Upsilon = [-\infty, \infty)^N$. Since we require that individual positions do not lead to default losses, we enforce condition $w^n(e^{\beta_{0|n}} - 1) \geq -1$ a.s., which implies that $w^n \in [0, 1]$ if f_{β_0} has full support over Υ. Moreover, in this case, we also require $\sum_{n=1}^{N} w^n \leq 1$, as otherwise the entire portfolio defaults. Clearly, Merton's original approach is rather limited when studying default risk, since leveraged positions (neither long nor short) are not allowed. In the sequel, we introduce some modifications to avoid overstating the risk of default. First, we define the random variables $Z_w = \sum_{n=1}^{N} w^n \alpha_{0|n}$ and $Y_w = \ln(1 + Z_w)$. Merton's approach then reads

$$\mu_{0|w}(t) = b_{0|w}(t) - \frac{1}{2}\sigma^2_{0|w}(t) + \lambda(t) \int_{[-\infty,\infty)} h\left(e^y - 1\right) f_{Y_w}(y, t) dy, \tag{29}$$

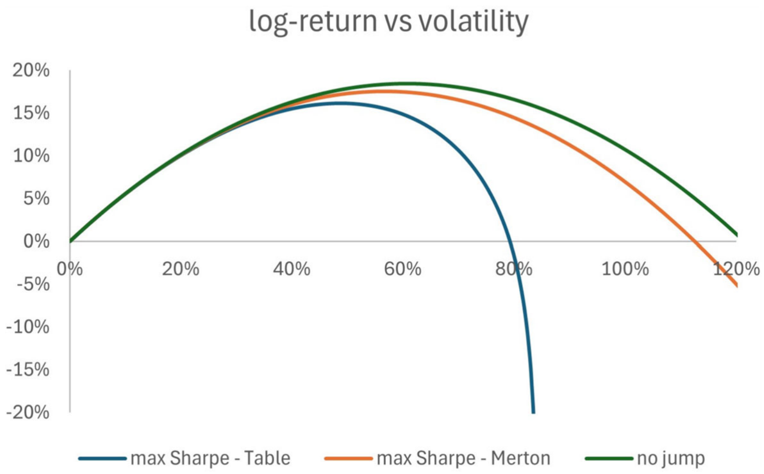

Fig. 4 This figure shows the rate of excess logarithmic return $\mu_{0|w}$ as a function of the volatility $\sigma_{0|w}$ for the maximal Sharpe strategies $w = k\hat{w}$, where k is used to generate various combinations. The plot compares the jump specification in Table 1 (*fearing a crash*) with Merton's approach. We also include the case of no jumps, that is where the fixed covariance matrix V_0 is fully allocated to the continuous part V_0^c

where f_{Y_w} is the density of Y_w. We now assume that Y_w has a Gaussian distribution and plot in Fig. 4 the rate of excess logarithmic return versus the volatility, for $w = k\hat{w}$, based on the case study: *fearing a crash* (Table 1). More precisely, we let $Y_{k\hat{w}} = a_k + b_k\varepsilon$, where ε is a standard Gaussian random variable (mean zero and unit variance), and analytically calibrate the parameters (a_k, b_k, λ), one calibration for each value of k, by matching the first moments of the relative jumps

$$\lambda \int_{-\infty}^{\infty} \left(e^{a_k + b_k \upsilon} - 1\right)^p \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} d\upsilon = \sum_{\upsilon=1}^{3} \left(\sum_{n=1}^{2} w^n \alpha_{0|n}^{\upsilon} \right)^p \lambda_{\upsilon}, \quad p \in \{0, 1, 2\}.$$

Although there might be more sophisticated ways to calibrate the parameters, it does not change the observation, from Fig. 4, that the jumps have only little impact in our version of Merton's approach. The explanation is that the choice of distribution ignores the point mass at $Z_w = -1$ (or $Y_w = -\infty$). The higher the leverage (i.e. the larger the value of k), the more likely it is that jumps in the primary assets make the wealth process go negative, thus leading to bankruptcy. For example, from Table 1, one sees that any leverage beyond the value $-1/\min_{\upsilon} \sum_n \hat{w}^n \alpha_{0|n}^{\upsilon} = 1.39$ makes the investor lose all his money (or more) eventually. With the current interpretation of Merton's approach, there is no such limit, and consequently the default risk can be largely underestimated (unless the calibrated Gaussian parameters are adjusted in some clever way). However, it is not trivial to model the point mass (representing default) so that it behaves well under leverage. For this reason, we much prefer the more direct approach outlined in Sect. 4, as further highlighted below.

5.1 Application to trading

When applying trading rules to the real market, there are several factors to consider that require the addition of jumps to a model. In particular, it is not feasible to trade in continuous-time, as markets are closed during the nights and weekends. In addition, fees are charged for every trade made. Thus, in reality, an investor executes trades at discrete time points. This results in observable finite price movements between non-trading intervals, which can be considered as jumps.

Henceforth, we consider an investor who trades at discrete times, with Δ years apart. For simplicity, we only consider trading in one risky asset. The information available to the investor, at each trading time $t_i = i\Delta$, is the distribution of the risky asset at the next trading time

$$F_{\beta_0}(\upsilon) = \mathbb{P}\left(\beta_{0|1} \leq \upsilon | \mathcal{F}_{t_i}\right), \quad \beta_{0|1} = \ln \frac{P_{0|1}(t_{i+1})}{P_{0|1}(t_i)}. \tag{30}$$

The investor applies a self-financing portfolio such that

$$\frac{X_{0|w}(t_{i+1}) - X_{0|w}(t_i)}{X_{0|w}(t_i)} = w \frac{P_{0|1}(t_{i+1}) - P_{0|1}(t_i)}{P_{0|1}(t_i)},$$

and calculates the rate of excess logarithmic return

$$\mu_{0|w} = \frac{1}{\Delta}\mathbb{E}\left[\ln \frac{X_{0|w}(t_{i+1})}{X_{0|w}(t_i)} | \mathcal{F}_{t_i}\right] = \frac{1}{\Delta}\mathbb{E}\left[\ln\left(1 + w\left(e^{\beta_{0|1}} - 1\right)\right) | \mathcal{F}_{t_i}\right].$$

This yields

$$\mu_{0|w} = b_{0|w} - \frac{1}{2}\sigma_{0|w}^2 + \lambda \int_{-\infty}^{\infty} h\left(w\left(e^{\upsilon} - 1\right)\right) f_{\beta_0}(\upsilon) d\upsilon, \tag{31}$$

with $\lambda = 1/\Delta$ and

$$b_{0|w} = \lambda \mathbb{E}\left[w\left(e^{\beta_{0|1}} - 1\right) | \mathcal{F}_{t_i}\right], \quad \sigma_{0|w}^2 = \lambda \mathbb{E}\left[\left(w\left(e^{\beta_{0|1}} - 1\right)\right)^2 | \mathcal{F}_{t_i}\right]. \tag{32}$$

The problem with Eq. (31), however, is precisely what we saw when analyzing Merton's original model, namely that we cannot study leveraged positions ($w > 1$ or $w < 0$) if the density f_{β_0} has full support on the real line. The way we proceed is to cast the setting into one driven by a marked point process. To explain the methodology, we set

$$F_{\beta_0}(\upsilon) = N\left(\frac{\upsilon - \mu_{0|1}\Delta}{\sigma_{0|1}\sqrt{\Delta}}\right), \tag{33}$$

where N denotes the distribution function of a standard Gaussian random variable, and recall the variable $\alpha_{0|1} = \exp(\beta_{0|1}) - 1$. We then assume that the mark space contains seven points, $\Upsilon = \{\upsilon_{-3}, \ldots, \upsilon_3\}$. For each discrete event in Υ we specify the relative jump size $\alpha^i_{0|1}$, $i \in \{-3, \ldots, 3\}$, according to

$$\alpha^i_{0|1} = e^{\mu_{0|1}\Delta + i\sigma_{0|1}\sqrt{\Delta}} - 1, \quad i \in \{-3, -2, -1, 0, 1, 2, 3\}. \tag{34}$$

The associated intensities, λ_i, are defined in terms of the corresponding probabilities p_i, such that $\lambda_i = \lambda p_i = p_i/\Delta$. We now set

$$\begin{aligned} p_0 &= N(c) - N(-c), \\ p_i &= N(c+i) - N(c+i-1), \quad i \in \{1, 2\}, \\ p_3 &= 1 - N(c+2), \\ p_{-i} &= p_i, \end{aligned}$$

so that $\sum_i p_i = 1$. Finally, we determine the free constant c according to

$$\sum_{i=-3}^{3} \alpha^i_{0|1}\lambda_i = \lambda \mathbb{E}\left[e^{\beta_{0|1}} - 1\right] = \frac{1}{\Delta}\left(e^{\mu_{0|1}\Delta + \frac{1}{2}\sigma^2_{0|1}\Delta} - 1\right). \tag{35}$$

In Fig. 5, we plot the rate of excess logarithmic return, which, after simplifications, takes the form

$$\mu_{0|w} = \sum_{i=-3}^{3} \ln(1 + w\alpha^i_{0|1})\lambda_i, \tag{36}$$

against the level of the trading strategy w^1 for various choices of the trading frequency $\lambda = 1/\Delta$. Here, the parameters used are set to match S&P500. When trading in continuous-time ($\Delta \to 0$), we know that $\mu_{0|k\hat{w}} = \frac{1}{2}k(2-k)s^2_{0|\hat{w}}$ and $\sigma^2_{0|k\hat{w}} = k^2 s^2_{0|\hat{w}}$. In this case, leveraging beyond the growth optimal Kelly strategy generates inefficient trading strategies in the sense that the same rate of excess logarithmic return can be achieved at a lower volatility. However, with discrete trading, the situation is more dire. For example, a leveraged investor that only trades every quarter (or less frequently) faces a big increase in default risk. We also note that for proper fractional Kelly strategies, such as the famous half-Kelly strategy $w = \frac{1}{2}\hat{w}$, the plot does not indicate any disadvantage in trading infrequently. However, this observation should be treated with care, as a different assumption for the distribution function in Eq. (33) could well increase the default risk; recall that with Merton's original approach, the upper limit of w equals 1. Another situation where we can expect a higher default risk is for strategies trading in more than one risky asset; especially if the investor is long some assets and short others. The reason being simply that in this case, the investor is exposed to both the left and right tails of the asset distributions.

It may sound like a major restriction to limit trading to discrete points in time; for example, when the trading interval is as long as six months. For this reason, we present an alternative interpretation of Fig. 5. First, let us recall that (similar to

Table 5 This table shows various jump-diffusion models that all agree on the total variance V_0 and the instantaneous Sharpe ratio $s_{0|1}$. Here, the mark space contains just a single point, with parameters $\alpha_{0|1}^{-3}$ and λ_{-3}

$\mu_{0\|1}$	$\sigma_{0\|1}$	$\alpha_{0\|1}^{-3}$	λ_{-3}	years between jumps	corresponding Δ
10.0%	20.0%	0.0%	∞	0	0
10.1%	19.6%	−7.8%	0.280	3.6	1 week
10.1%	19.6%	−15.2%	0.065	15.5	1 month
10.1%	19.7%	−24.0%	0.022	46.4	3 months
10.0%	19.7%	−31.2%	0.011	92.8	6 months

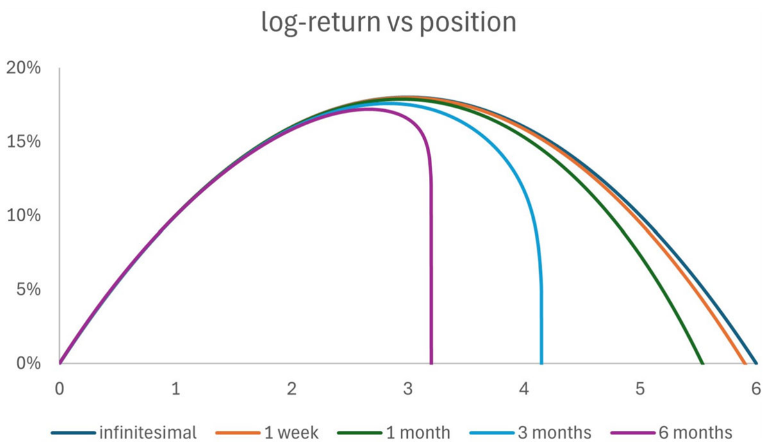

Fig. 5 This figure shows the rate of excess logarithmic return $\mu_{0|w}$ as a function of the strategy w, for various levels of the trading period Δ. The parameters used are: $\mu_{0|1} = 0.1$ and $\sigma_{0|1} = 0.2$, such that $\hat{w} = 1/2 + \mu_{0|1}/\sigma_{0|1}^2 = 3$ as $\Delta \to 0$. Furthermore, the calibrated parameter $c \approx 0.55$

Merton's approach) default is related to the worst-case relative jump size, which in our setting equals $\alpha_{0|1}^{-3}$. In Table 5, we present various jump-diffusion models that keep the total variance $\sigma_{0|1}^2$ and the instantaneous Sharpe ratio $s_{0|1}$ unchanged. The models use a reduced mark space $\bar{\Upsilon} = \{\upsilon_{-3}\}$, containing the worst-case point from the original mark space, with parameters $\alpha_{0|1}^{-3}$ and λ_{-3}. Thereafter, the asset parameters are modified, using Eqs. (10) and (16), according to

$$\mu_{0|1} \to \mu_{0|1} + \left(h\left(\alpha_{0|1}^{-3}\right) + \frac{1}{2}\left(\alpha_{0|1}^{-3}\right)^2 \right) \lambda_{-3}, \quad \sigma_{0|1}^2 \to \sigma_{0|1}^2 - \left(\alpha_{0|1}^{-3}\right)^2 \lambda_{-3}.$$

What is interesting to note is that the models presented in Table 5 generate excess logarithmic return rates, $\mu_{0|w}$, which almost perfectly match those in Fig. 5. This alternative interpretation agrees well with historical data for the largest relative losses of the S&P500 index. It also shows that the return of a trading strategy is mainly driven by the diffusion parameters and the worst-case jump (represented by its size and intensity); an observation that further strengthens the practical relevance of Example 1 and 2. Only in rather extreme situations, for example the case study *picking a winner*, must we also account for the best-case jump.

We end the discussion by emphasizing that trading is discrete by nature and that related properties (like sensitivity to trading frequency) should always be evaluated, especially when leverage is applied. The framework of marked point processes is well suited for such robustness analysis and for the corresponding worst-case jump analysis.

6 Comments on volatility

In this Section, we take a closer look at the instantaneous covariance matrix and how best to define a measure of volatility in the presence of jumps. What are the implications if we do not follow Björk and Slinko (2006) and Christensen and Platen (2007) when defining volatility?

Our first observation is that, with jumps absent, the volatility can be expressed in either of the two ways:

$$\sigma^2_{0|w}(t) = \frac{1}{X^2_{0|w}(t-)}\frac{d}{dt}\langle X_{0|w}, X_{0|w}\rangle(t) = w^i(t)V^c_{0|i,j}(t)w^j(t), \tag{37}$$

$$\tilde{\sigma}^2_{0|w}(t) = \frac{d}{dt}\langle \ln X_{0|w}, \ln X_{0|w}\rangle(t) = w^i(t)V^c_{0|i,j}(t)w^j(t). \tag{38}$$

However, with jumps present, these two expressions are no longer identical. Using Eqs. (10), (11), and Lemma 1, we see that

$$\sigma^2_{0|w}(t) = w^i(t)\left(V^c_{0|i,j}(t) + \int_\Upsilon \alpha_{0|i}(\upsilon,t)\alpha_{0|j}(\upsilon,t)\lambda(d\upsilon,t)\right)w^j(t), \tag{39}$$

$$\tilde{\sigma}^2_{0|w}(t) = w^i(t)V^c_{0|i,j}(t)w^j(t) + \int_\Upsilon \ln^2\left(1 + w^n(t)\alpha_{0|n}(\upsilon,t)\right)\lambda(d\upsilon,t), \tag{40}$$

such that $\sigma_{0|kw} = |k|\sigma_{0|w}$ but $\tilde{\sigma}_{0|kw} \neq |k|\tilde{\sigma}_{0|w}$, for any real-valued predictable process k. Consequently, what we call, the logarithmic volatility $\tilde{\sigma}$ is not suitable when defining the instantaneous Sharpe ratio (assuming that we want this quantity to be leverage invariant). For small jumps, though, the expressions are close in the sense that

$$\tilde{\sigma}^2_{0|w}(t) = \sigma^2_{0|w}(t) + \int_\Upsilon \mathcal{O}\left(w^n(t)\alpha_{0|n}(\upsilon,t)\right)^3\lambda(d\upsilon,t). \tag{41}$$

However, for finite time horizons, it is the logarithmic volatility that is of importance, since it is to the logarithmic returns that the law of large numbers (and the central limit theorem) apply. In Figs. 6 and 7, we plot the behavior of the rate of excess logarithmic return $\mu_{0|w}$ against the logarithmic volatility $\tilde{\sigma}_{0|w}$ for the two case studies in

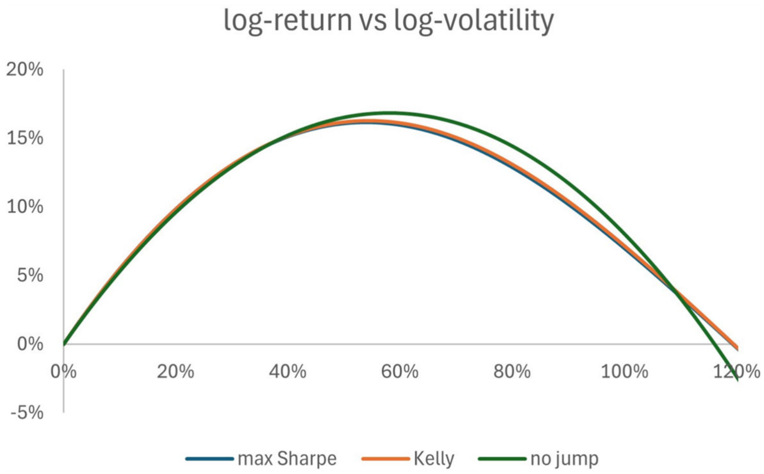

Fig. 6 This figure shows the rate of excess logarithmic return $\mu_{0|w}$ as a function of the logarithmic volatility $\tilde{\sigma}_{0|w}$ for the strategies: $w = k\hat{w}$ (maximal Sharpe) and $w = kw_*$ (Kelly), with k used to generate various combinations. The jump specification is given by Table 1 (*fearing a crash*). We also compare to a situation with no jumps, that is where the modified covariance matrix $\tilde{V}_0$ is fully allocated to the continuous part V_0^c

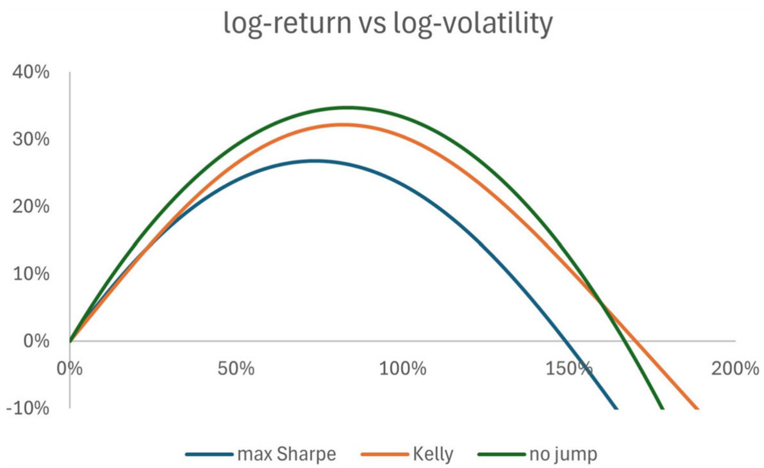

Fig. 7 This figure shows the rate of excess logarithmic return $\mu_{0|w}$ as a function of the logarithmic volatility $\tilde{\sigma}_{0|w}$ for the strategies: $w = k\hat{w}$ (maximal Sharpe) and $w = kw_*$ (Kelly), with k used to generate various combinations. The jump specification is given by Table 3 (*picking a winner*). We also compare to a situation with no jumps, that is where the modified covariance matrix $\tilde{V}_0$ is fully allocated to the continuous part V_0^c

Sect. 4. In order to compare with the situation where jumps are not present, we further set

$$\tilde{V}_{0|v,w}(t) = v^i(t)\tilde{V}_{0|i,j}(t)w^j(t), \quad \tilde{V}_{0|i,j}(t) = V^c_{0|i,j}(t) + \tilde{V}^d_{0|i,j}(t), \tag{42}$$

Table 6 This table shows the volatility and the logarithmic volatility for the growth optimal maximal Sharpe and Kelly strategies. Note that the volatilities are lower (higher) than the logarithmic volatilities for the case study: *fearing a crash* (*picking a winner*)

Case study	$\sigma_{0\|k_*\hat{w}}$	$\sigma_{0\|w_*}$	$\tilde{\sigma}_{0\|k_*\hat{w}}$	$\tilde{\sigma}_{0\|w_*}$
fearing a crash	48.7%	49.7%	53.9 %	54.4%
picking a winner	81.5%	119.1%	73.6 %	82.1%

with

$$\tilde{V}^d_{0|i,j}(t) = \int_\Upsilon \ln(1+\alpha_{0|i}(\upsilon,t))\ln(1+\alpha_{0|j}(\upsilon,t))\lambda(d\upsilon,t). \tag{43}$$

We then assign the full covariance matrix $\tilde{V}_0$ to its continuous part while setting the discrete part to zero. Clearly, this methodology is an approximate attempt to make the instantaneous covariance matrices comparable. However, as can be seen from the figures, it agrees well with, in particular, Kelly's approach. Hence, despite the many attractive features of the maximal Sharpe strategy, we argue that this approach might lead to suboptimal allocations (especially when large positive jumps are present) due to the non-linearity of the logarithmic volatility with respect to leverage. In contrast, we see that Kelly's approach deviates less from known results obtained by studying portfolio allocations in markets modeled without jumps.

In Table 6 we provide a summary related to the two volatility measures for each of the case studies in Sect. 4. Although it is easy to predict which of the measures is higher in each separate case, it is somewhat surprising to see such a significant reduction in the logarithmic volatility for the Kelly strategy when large positive jumps dominate. In general, it is difficult to say that one volatility measure is always better than the other because the answer very much depends on the context. For example, when addressing optimal long-term growth, it makes sense to focus on logarithmic volatility. However, for short time periods, the first volatility measure is more natural, since it connects better with the self-financing property. Below, we introduce yet another candidate

$$\bar{\sigma}^2_{0|w}(t) = \frac{1}{3}\sigma^2_{0|w}(t) + \frac{2}{3}\tilde{\sigma}^2_{0|w}(t). \tag{44}$$

The reason for this choice follows from Eq. (16), which then takes the form

$$\mu_{0|w}(t) = b_{0|w}(t) - \frac{1}{2}\bar{\sigma}^2_{0|w}(t) + \int_\Upsilon \bar{h}\left(w^n(t)\alpha_{0|n}(\upsilon,t)\right)\lambda(d\upsilon,t), \tag{45}$$

where now

$$\bar{h}(x) = \ln(1+x) - x + \frac{1}{6}x^2 + \frac{1}{3}\ln^2(1+x) \approx \frac{1}{18}x^4 - \frac{7}{90}x^5 + \dots \tag{46}$$

Since $\bar{h}$ almost vanishes for values close to zero, we obtain an expression for the rate of excess logarithmic return that essentially looks as if derived from a pure diffusion model. We conclude the Section by acknowledging that it is not obvious how best to define a volatility measure when jumps are present.

7 Conclusions

In this paper we show that when asset prices have jumps, there is a distinction between maximal Sharpe and Kelly strategies. The result should be compared with a pure diffusion model, where no such distinction is made. We show that the asset allocations between a jump-diffusion and a pure diffusion model can change considerably. Our results further indicate that a Kelly trader fears market crashes but favors stock picking more than a maximal Sharpe trader. However, in terms of local characteristics, the discrepancies manifest themselves mainly for highly volatile trading strategies. This means that maximal Sharpe strategies are good approximations of most Kelly strategies. Yet, when comparing the qualitative behavior to the situation without jumps, we find that Kelly's approach coincides more closely. We also explain why Merton's approach to jump modeling can either largely overestimate or underestimate the risk of bankruptcy, and further introduce a framework better suited for answering such questions. In particular, we find that the return of a trading strategy is mainly driven by the diffusion parameters and the worst-case jump (represented by its size and intensity). Only in extreme situations must we also account for the best-case jump.

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Declarations

Conflict of Interest Both authors work at Hilbert Group, an investment manager. The authors have no relevant financial or non-financial interests to disclose.

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